2014/2 ENGINEERING DEPARTMENTS PHYSICS 2
RECITATION 7
(FARADAY'S LAW-INDUCTANCE)

1. A conducting rod of length L is free to slide on two parallel conducting bars, as shown in **Figure 1**. Two resistors R_1 and R_2 are connected across the ends of the bars to form a loop. A constant magnetic field B is directed perpendicular into the page. An external agent pulls the rod to the left with a constant speed of $\vec{v}$. Find

a) the currents in both resistors,

b) the total power delivered to the resistance of the circuit, and

c) the magnitude of the applied force that is needed to move the rod with this constant velocity.

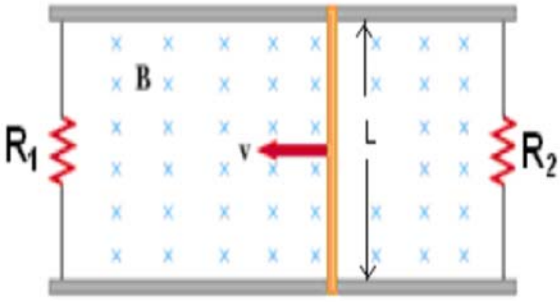
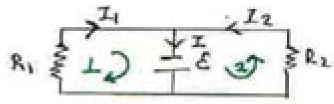
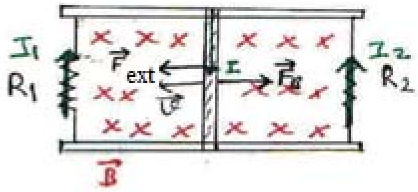


Figure 1



1) $\mathcal{E} - I_1 R_1 = 0$

2) $\mathcal{E} - I_2 R_2 = 0$



a) $$\mathcal{E} = -\frac{d\phi_B}{dt} = -BLv$$

$$\left.\begin{aligned} I_1 &= \frac{|\mathcal{E}|}{R_1} = \frac{BLv}{R_1} \\ I_2 &= \frac{|\mathcal{E}|}{R_2} = \frac{BLv}{R_2} \end{aligned}\right\}$$

b) $$P_R = I_1|\mathcal{E}| + I_2|\mathcal{E}| = \frac{\mathcal{E}^2}{R_{eq.}}$$

$$= (I_1 + I_2)|\mathcal{E}| = \mathcal{E}^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$\boxed{P_R = B^2L^2v^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$$

c) $$I = I_1 + I_2 \quad ; \quad \vec{F}_B = I\,\vec{L} \times \vec{B}$$

$$F_B = ILB = |\mathcal{E}|LB\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$\boxed{F_B = B^2L^2v\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$$ to the right

An external agent must then exert a force to the left to keep the bar moving with constant speed.

$$\vec{F}_B + \vec{F}_{ext} = 0 \rightarrow \boxed{\vec{F}_{ext} = -\vec{F}_B}$$

2. **Figure 2** shows a rod of length $L = 10\ cm$ that is forced to move at constant speed $v = 5\ m/s$ along horizontal rails. The rod, rails, and connecting strip at the right form a conducting loop. The rod has resistance $R = 0.4\ \Omega$; the rest of the loop has negligible resistance. A current $I = 100\ A$ through the long straight wire at distance $a = 10\ mm$ from the loop sets up a (nonuniform) magnetic field through the loop. Find the

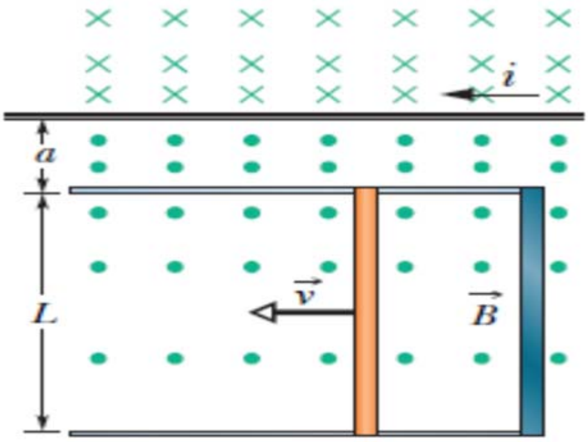
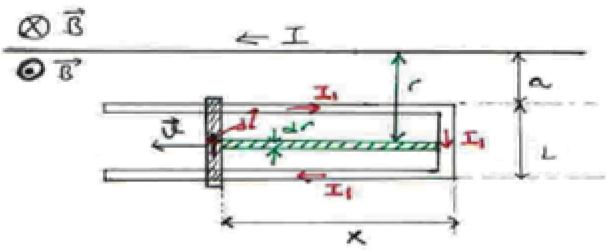


Figure 2

- **a)** emf and
- **b)** current induced in the loop.
- **c)** At what rate is thermal energy generated in the rod?
- **d)** What is the magnitude of the force that must be applied to the rod to make it move at constant speed?
- **e)** At what rate does this force do work on the rod?

a)



$dA = x\,dr$

the magnetic field of the long straight wire is

$$B = \frac{\mu_0 I}{2\pi r}$$

where r is the distance from the wire.

Its direction is out of the rod-rail plane ⊙

$$\phi_B = \int \vec{B}\cdot d\vec{A} = \int B\,dA = \frac{\mu_0 I x}{2\pi}\int_a^{a+L}\frac{dr}{r}$$

$$\boxed{\phi_B = \frac{\mu_0 I x}{2\pi}\ln\left(\frac{a+L}{a}\right)}$$

$$\mathcal{E} = -\frac{d\phi_B}{dt} = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{a+L}{a}\right)\left[\frac{dx}{dt}\right] \rightarrow v$$

$$\mathcal{E} = -\frac{4\pi\cdot 10^{-7}\cdot 100\cdot 5}{2\pi}\ln\left(\frac{1{,}0+10}{1{,}0}\right) = \underline{\underline{-0{,}24\ mV}}$$

b)

$$I_{ind.} = \frac{\mathcal{E}}{R} = \frac{0{,}24\cdot 10^{-3}}{0{,}40} = \boxed{0{,}60\ mA}$$

c)

$$P_{thermal} = I_{ind}^2 R = (6\cdot 10^{-4})^2\cdot 0{,}40 = 0{,}14\ \mu W$$

d)

$$\int d\vec{F}_B = \int I_{ind}\, d\vec{l}\times\vec{B} \rightarrow \vec{F}_B = \int I_{ind}\, d\vec{l}\times\vec{B}$$ (to the right)

$$F_B = \int_{L+a}^{a} I_{ind}\frac{\mu_0 I}{2\pi r}\,dl \rightarrow dl = -dr$$ so,

$$= -\frac{\mu_0 I I_{ind}}{2\pi}\int_{L+a}^{a}\frac{dr}{r} = \frac{\mu_0 I I_{ind}}{2\pi}\int_a^{L+a}\frac{dr}{r}$$

$$\boxed{F_B = \frac{\mu_0 I_{nd} \cdot I}{2\pi} \ln\left(\frac{L+a}{a}\right)}$$

$$\vec{F}_{\text{external}} + \vec{F}_B = 0 \Rightarrow \vec{F}_{\text{ext.}} = -\vec{F}_B \quad \text{to the left}$$

$$\vec{F}_{\text{ext.}} \parallel \vec{v}$$

$$\boxed{F_{\text{external}} = 2{,}88 \cdot 10^{-8}\ N}$$

e) the external agent does work at the rate

$$P = \vec{F}_{\text{ext.}} \cdot \vec{v}$$

$$\boxed{P = 2{,}88 \cdot 10^{-8} \cdot 5 = 1{,}44 \cdot 10^{-7}\ W}$$

3. A long solenoid has *n=400 turns per meter* and carries a current given by $I = 30(1 - e^{-1.6t})\ (A)$. Inside the solenoid and coaxial with it is a coil that has a radius of *6 cm* and consists of a total of *N= 250 turns* of fine wire (**Figure 3**). What emf is induced in the coil by the changing current?

($\mu_0 = 4\pi x 10^{-7} Wb/A.m$).

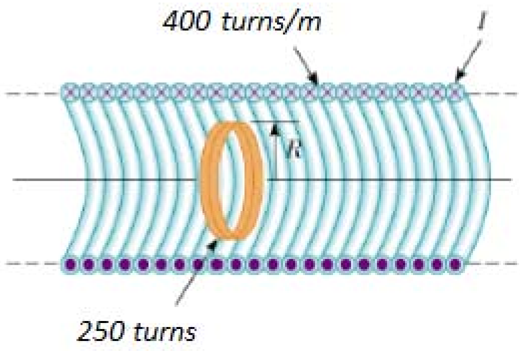
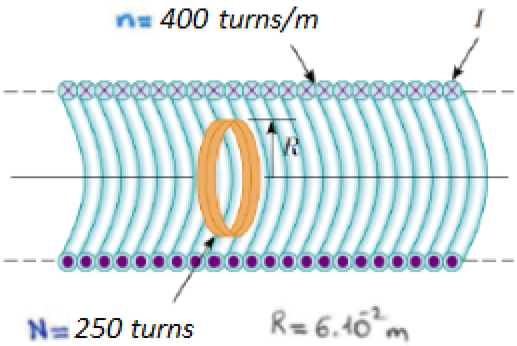


Figure 3

The magnetic field along the axis of the solenoid:



$$B = \mu_0 n I$$

$$B = 4\pi.10^{-7}.400.30.(1 - e^{-1,6t})$$

$$B = 1,5.10^{-2}(1 - e^{-1,6t}) \quad (T)$$

$$\Phi_B = \int \vec{B}.d\vec{A} = \int B\,dA\cos 0^\circ$$

$$\Phi_B = \int_0^R 1,5.10^{-2}(1 - e^{-1,6t})(2\pi r\,dr)$$

$$\Phi_B = 1,5.10^{-2}(1 - e^{-1,6t}).2\pi \int_0^{6.10^{-2}} r\,dr = 1,5.10^{-2}.(1 - e^{-1,6t})2\pi\left[\frac{r^2}{2}\right]_0^{6.10^{-2}}$$

$$\Phi_B = 1,7.10^{-4}(1 - e^{-1,6t}) \quad (Wb)$$ (The flux through the solenoid)

As the solenoid flux is changing with time, the induced emf in the coil;

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$

$$\mathcal{E} = -250.\frac{d}{dt}\left[1,7.10^{-4}(1 - e^{-1,6t})\right]$$

$$\mathcal{E} = -250.1,7.10^{-4}.1,6.e^{-1,6t}$$

$$\mathcal{E} = -6,8.10^{-2}.e^{-1,6t} \quad (V)$$

$$\boxed{\mathcal{E} = -68.e^{-1,6t} \quad (mV)}$$

4. For the situation shown in **Figure 4**, the magnetic field changes with time according to the expression $B = (2t^3 - 4t^2 + 1)\ T$ and *r=2R=5cm*.

a) Calculate the magnitude and direction of the force exerted on an electron located at point P when *t=2s.*

b) At what time is this force equal to zero?

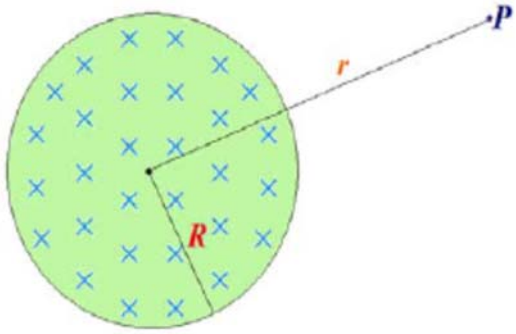


Figure 4

a)

$$\Phi_B = \vec{B}.\vec{A} = BA\cos 0°$$

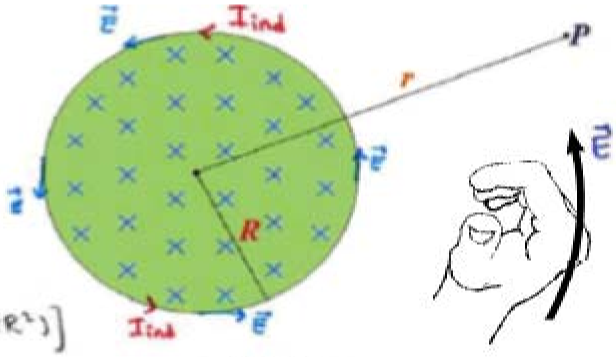
$$\Phi_B = (2t^3 - 4t^2 + 1)(\pi R^2)$$

$$\oint \vec{E}.d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$E.(2\pi r) = -\frac{d}{dt}\left[(2t^3-4t^2+1)(\pi R^2)\right]$$

$$E.(2\pi r) = -(\pi R^2)(6t^2 - 8t)$$

$$E = -\frac{R^2}{2r}(6t^2 - 8t)$$



$r = 2R = 5cm$

At t=2s, $E = -\frac{2,5.10^{-2}}{4}(6.2^2 - 8.2)$

$$E = -0,05\ (V/m)$$

$$F = -eE$$

$$F = (-1,6.10^{-19}).(-0,05)$$

$\boxed{F = 8.10^{-21}\ (N)}$ (clockwise)

b) $E = 0 \Rightarrow F = 0$

$$\frac{d\Phi_B}{dt} = 0 \quad ; \quad \frac{dB}{dt} = 0$$

$$\frac{d}{dt}(2t^3 - 4t^2 + 1) = 0$$

$$6t^2 - 8t = 0$$

$$t = \frac{4}{3}$$

$$\boxed{t = 1,33\ (s)}$$

5. An *820-turn* wire coil of resistance *24 Ω* is placed around a *12500-turn* solenoid *7 cm* long, as shown in **Figure 5**. Both coil and solenoid have cross-sectional areas of 10^{-4} m^2.

a) How long does it take the solenoid current to reach *63.2%* of its maximum value?

Determine

b) the average back emf caused by the self-inductance of the solenoid during this time interval,

c) the average rate of change in magnetic flux through the coil during this time interval, and

d) the magnitude of the average induced current in the coil.

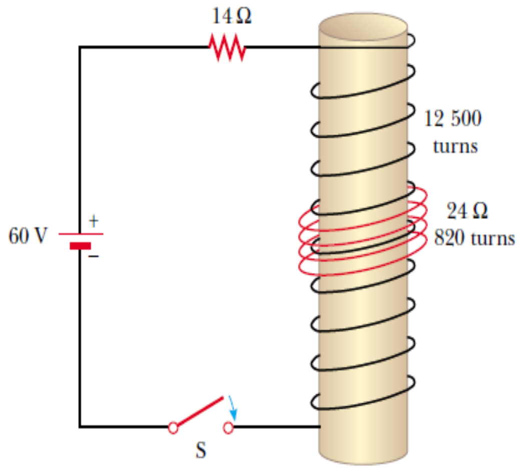
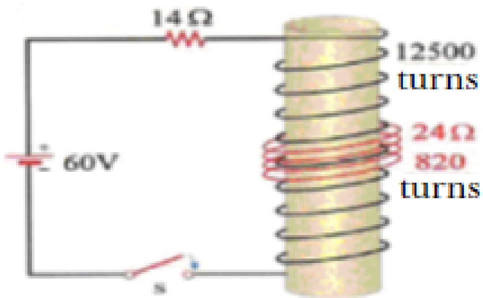
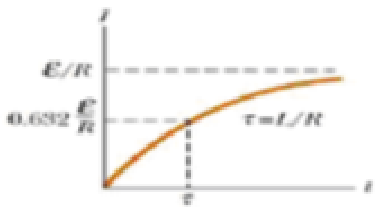


Figure 5





a) Inductance of a coil having N turns;

$$L = \frac{N\Phi_B}{I}$$

The magnetic field of the solenoid

$$L = \frac{N}{I} B.A = \frac{N}{I} \mu_0 \frac{N}{\ell} I . A$$

$$L = \mu_0 \frac{N^2 A}{\ell}$$

$$L = 4\pi.10^{-7} \cdot \frac{(12500)^2 \cdot (10^{-4})}{70.10^{-2}}$$

$$L = 0,28 \text{ (H)}$$

$$\tau = \frac{L}{R}$$

$$\tau = \frac{0,28}{14}$$

$$\boxed{\tau = 20 \text{ (ms)}}$$

b) $$\varepsilon_L = -L \frac{dI}{dt}$$

$$|\bar{\varepsilon}_L| = L\left(\frac{\Delta I}{\Delta t}\right) = L\left(\frac{I_f - I_i}{t_f - t_i}\right)$$

$$|\bar{\varepsilon}_L| = 0,28 \cdot \left(\frac{2,74}{20.10^{-3}}\right)$$

$$\boxed{|\bar{\varepsilon}_L| \cong 38 \text{ (V)}}$$

$I_i = 0$

$I_f = 0,632 . I_{max}$

$I_f = 0,632 . \frac{\Delta V}{R} = 0,632 . \frac{60}{14}$

$I_f = 2,74$ A

$t_i = 0$

$t_f = 20$ ms

c) The average rate of change of flux through each turn of the overwrapped concentric coil is the same as that through a turn on the solenoid:

$$\frac{\Delta\Phi_B}{\Delta t} = \frac{\Delta(B.A)}{\Delta t} = \frac{\Delta(\mu_0 \frac{N}{\ell} I . A)}{\Delta t} = \mu_0 \frac{N.A}{\ell} \frac{\Delta I}{\Delta t}$$

$$\frac{\Delta\Phi_B}{\Delta t} = 4\pi.10^{-7} . \frac{12\,500 . 10^{-4}}{70.10^{-2}} . \frac{2,74}{20.10^{-3}} \cong 3.10^{-3} \text{ V} \; ; \; \boxed{\frac{\Delta\Phi_B}{\Delta t} = 3 \text{ (mV)}}$$

d) $$|\varepsilon_L| = N \frac{\Delta\Phi_B}{\Delta t}$$

$$I = \frac{|\varepsilon_L|}{R} = \frac{N}{R} \frac{\Delta\Phi_B}{\Delta t}$$

$$I = \frac{820}{24} . 3.10^{-3}$$

$$I \cong 0,103 \text{ A} \; ; \; \boxed{I = 103 \text{ (mA)}}$$

6. The toroid in **Figure 6** consists of *N* turns and has a rectangular cross section. Its inner and outer radii are *a* and *b*, respectively.

a) Show that the inductance of the toroid is

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$$

b) Using this result, compute the self-inductance of a *500*-turn toroid for which *a=10cm*, *b=12cm* and *h=1cm*.

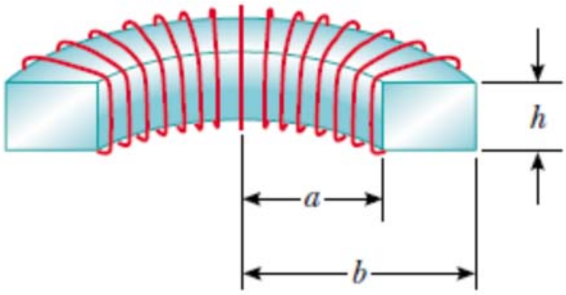
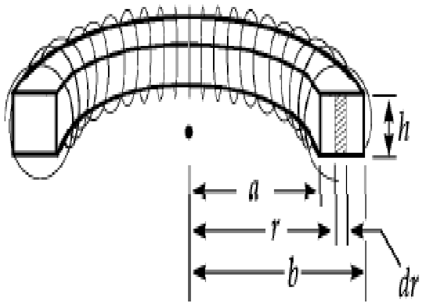


Figure 6

$$B = \frac{\mu_0 N I}{2\pi r} \quad \text{toroid}$$



$$dA = h\,dr$$

(a) $$\Phi_B = \int B\,dA = \int_a^b \frac{\mu_0 N I}{2\pi r} h\,dr = \frac{\mu_0 N I h}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 N I h}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$L = \frac{N\Phi_B}{I} = \boxed{\frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)}$$

(b) $$L = \frac{\mu_0 (500)^2 (0.010\,0)}{2\pi} \ln\left(\frac{12.0}{10.0}\right) = \boxed{91.2\ \mu\text{H}}$$

7. In **Figure 7**, the battery is ideal and $\varepsilon = 10V$, $R_1 = 5\Omega$, $R_2 = 10\Omega$ and $L = 5H$. Switch *S* is closed at time *t=0*. Just afterwards and a long time later, what are

a) the current I_1 through the resistor 1,

b) the current I_2 through the resistor 2,

c) the current I through the switch,

d) the potential difference V_2 across resistor 2,

e) the potential difference V_L across the inductor.

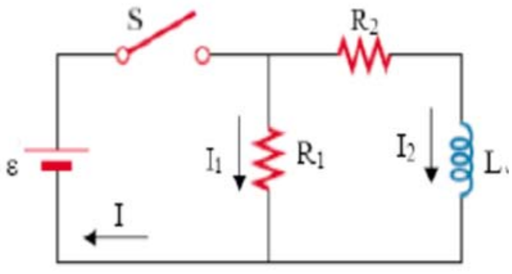
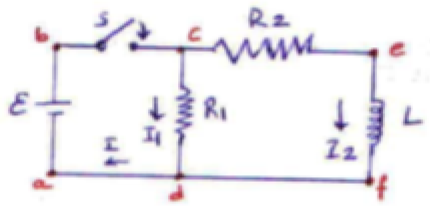


Figure 7



When switch *S* is just closed (at t=0)

At junction c

$$\boxed{I = I_1 + I_2}$$

$$\boxed{I_2 = 0} \rightarrow \boxed{I = I_1}$$

(a b c d a) loop;

$$\varepsilon - I_1 R_1 = 0 \rightarrow I_1 = \frac{\varepsilon}{R_1} = \frac{10}{5} = \underline{2A}$$

$$\underline{I = I_1 = 2A}$$

at $t \rightarrow \infty$;

$$I = I_1 + I_2$$

(a b c d a) loop $\Rightarrow$ $\varepsilon - I_1 R_1 = 0 \rightarrow I_1 = \frac{\varepsilon}{R_1} = \frac{10}{5} = \underline{2A}$

(a b e f a) loop $\Rightarrow$ $\varepsilon - I_2 R_2 = 0 \rightarrow I_2 = \frac{\varepsilon}{R_2} = \frac{10}{10} = \underline{1A}$

$$I = I_1 + I_2 = 2 + 1 = \underline{3A}$$

d) at $t = 0 \rightarrow V_2 = I_2 \cdot R_2 = 0 \cdot 10 = 0$

at $t \rightarrow \infty \rightarrow V_2 = I_2 \cdot R_2 = 1 \cdot 10 = 10V$

e) (c e f d c) loop;

$$\boxed{V_L = I_2 R_2 - I_1 R_1}$$

at $t = 0 \rightarrow V_L = 0 \cdot 10 - 2.5 = \underline{-10 V}$

at $t = \infty \rightarrow V_L = 0$ As I_2 is constant

8. The switch *S* is closed at *t=0* in the *RL* circuit as shown in **Figure 8**.

a) Find I_1, I_2 and I_3 currents when the switch *S* is closed.

b) Find I_1, I_2 and I_3 currents after the switch *S* has been closed for a length of time sufficiently long.

c) What is potential difference through the *resistor 2* when the switch *S* is opened (*t=0*) again after being closed for a long time?

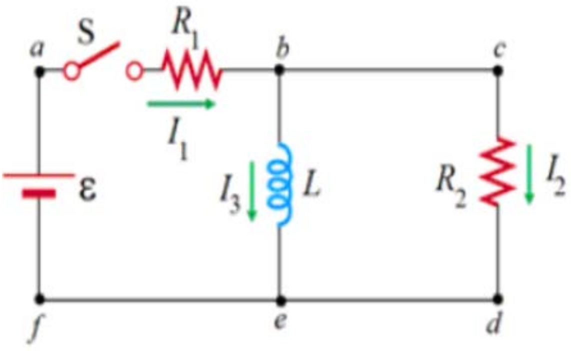
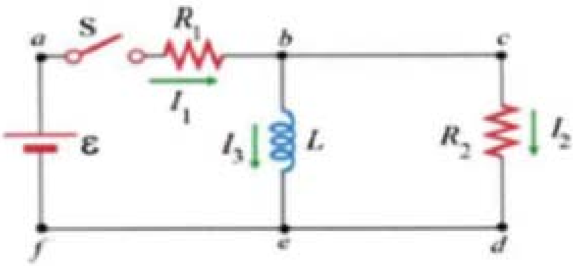


Figure 8

a) $\mathcal{E}_L = -L \frac{dI}{dt}$



at $t=0$ $\boxed{I_3 = 0}$

$I_1 = I_2$

acdfa loop : $-I_1R_1 - I_2R_2 + \mathcal{E} = 0$

$$\boxed{I_1 = I_2 = \frac{\mathcal{E}}{R_1+R_2}}$$

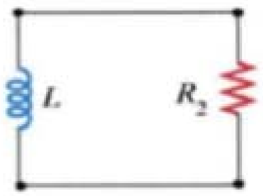
b) at $t \to \infty$ $\boxed{I_2 = 0}$

$I_1 = I_3$

abefa loop : $-I_1R_1 + \mathcal{E} = 0$

$$\boxed{I_1 = I_3 = \frac{\mathcal{E}}{R_1}}$$

c)



The energy stored in the inductor is consumed through the resistor 2.

$$IR_2 + L\frac{dI}{dt} = 0$$

$$I = I_0 e^{-t/\tau}$$

$$I_0 = \frac{\mathcal{E}}{R_1}$$

at $t=0$ $\Rightarrow$ $\Delta V_{R_2} = I_0 R_2 = \frac{\mathcal{E}}{R_1} \cdot R_2$

$$\boxed{\Delta V_{R_2} = \mathcal{E}}$$